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STABILITY OF THERMOCAPILLARY FLOW IN A FLAT LAYER WITH ALLOWANCE FOR THE SORET EFFECT

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The stability of thermocapillary two-component liquid flow is studied taking into account thermal diffusion. An explicit expression is obtained to construct neutral Marangoni numbers under the assumption of monotonicity of perturbations. The thermocapillary and hydrodynamic instability mechanisms are considered. It is shown that plane perturbations are the greatest hazard to the stability of return flow.

Key words: *stability, thermocapillary flow, thermal diffusion, Marangoni number, neutral curve.*

In studies of the thermocapillary effect on the behavior of a free-surface liquid in heat- and mass-transfer problems, emphasis is usually placed on the conditions under which the liquid becomes unstable and motion sets in [1–4]. The stability of similar motions induced by thermocapillary forces has been studied less extensively. Research in this direction was started in [5] and continued in [6, 7]. These papers studied the stability of thermocapillary flow of a flat homogeneous liquid layer under a longitudinal temperature gradient. Two solutions — linear and quadratic — were obtained and analyzed for stability under the assumptions of a rigid [6] and a deformed [7] free surface.

In the present paper, the stability of thermocapillary two-component liquid flow with a rigid free boundary is studied taking into account thermal diffusion (the Soret effect).

1. We consider a flat layer of a heat-conducting viscous liquid in the absence of mass forces. Let surfactants with a surface concentration $\Gamma(t,x,y)$ be concentrated at the free surface. The Navier–Stokes equation and the heat-convection and impurity-concentration equations taking into account thermal diffusion are written as

$$\frac{d\boldsymbol{u}}{dt} + \frac{1}{\rho}\,\nabla p = \nu\Delta\boldsymbol{u}, \qquad \operatorname{div}\boldsymbol{u} = 0,$$
$$\frac{d\theta}{dt} = \chi\Delta\theta, \qquad \frac{dc}{dt} = D\Delta\Big(c + \frac{k_T}{\theta_m}\,\theta\Big). \tag{1.1}$$

Here $\boldsymbol{u} = (u,v,w)$ is the liquid velocity, p is the pressure, θ is the temperature, c is the impurity concentration, ν and χ are the kinetic viscosity and thermal diffusivity, respectively, ρ is the density, D is the diffusion coefficient, k_TD is the thermal-diffusion coefficient, and θ_m is a certain average temperature.

We assume that transfer processes in the gas outside the liquid can be ignored. Let p_1 and θ_1 be the specified pressure and temperature of the gas on the free surface L. Then, on L, the following conditions should be satisfied:

$$(p_1 - p)I\boldsymbol{n} + 2\rho\nu D(\boldsymbol{u}) = 2\sigma H\boldsymbol{n} + \nabla_\tau\sigma,$$
$$\lambda\,\frac{\partial\theta}{\partial\boldsymbol{n}} + \beta(\theta-\theta_1) + Q = 0, \qquad f_t + \boldsymbol{u}\nabla f = 0. \tag{1.2}$$

Here $\boldsymbol{n}$ is the outward normal vector to L, H is the average free-surface curvature, I is the unit tensor, $\nabla_\tau = \nabla - (\boldsymbol{n}\nabla)\boldsymbol{n}$ is the surface gradient, λ and β are the thermal conductivity and the interfacial heat-transfer factor, Q is the heat flux through the free surface, and $f(x,t) = 0$ is the free-surface equation L.

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We assume that there is no mass flux through the rigid surface and the velocity and impurity concentration are subjected to the constraints

$$u = 0, \qquad \frac{\partial c}{\partial \boldsymbol{n}} + \frac{k_T}{\theta_m}\frac{\partial \theta}{\partial \boldsymbol{n}} = 0. \tag{1.3}$$

Let the surface-tension variation depends linearly on the temperature and surfactant concentration:

$$\sigma = \sigma_0 - æ(\theta - \theta_0) - \gamma(\Gamma - \Gamma_0).$$

We place the coordinate origin on the solid surface so that the x and y axes are directed horizontally and the z axis is directed vertically upward. The equations for the solid and free boundaries are $z = 0$ and $z = l$, respectively. The transfer of the surfactant along the free boundary is described by the equation

$$\frac{\partial \Gamma}{\partial t} + \frac{\partial}{\partial x}(u\Gamma) + \frac{\partial}{\partial y}(v\Gamma) - D_s\Big(\frac{\partial^2 \Gamma}{\partial x^2} + \frac{\partial^2 \Gamma}{\partial y^2}\Big) = j_n, \tag{1.4}$$

where u and v are the liquid velocity components on the surface, D_s is the surface-diffusion coefficient of the surfactant, and j_n is the mass flux from the surface into the bulk.

The flux j_n is determined by the surfactant transfer into the liquid, and taking into account thermal diffusion, it is written as

$$j_n = -D\Big(\frac{\partial c}{\partial z} + \frac{k_T}{\theta_m}\frac{\partial \theta}{\partial z}\Big) \qquad (z = l). \tag{1.5}$$

Here c is the concentration of the dissolved surfactant in the liquid.

On the other hand, the mass transfer between the surface and the liquid occurs by adsorption and desorption and the flux is given by

$$j_n = K_A c - K_D \Gamma \qquad (z = l), \tag{1.6}$$

where K_A and K_D are the adsorption and desorption coefficients, respectively.

Let a constant surfactant concentration $\Gamma = \Gamma_0$ be specified on the free surface. We consider the case where along the free boundary there is a temperature gradient $\theta_x\Big|_{z=l} = -A$.

We perform nondimensionalization. As the characteristic quantities we use $A\nu/\chi$ for the temperature, ν/l for the velocity, $\rho\nu^2/l^2$ for the pressure, l for the length, l^2/ν for time, Γ_0 for the surfactant concentration, and $K_D\Gamma_0/K_A$ for the impurity concentration.

The solution of problem (1.1)–(1.6) is sought in the form

$$u = u(\xi), \quad v = w = 0, \quad \theta = \theta(x, \xi), \quad c = c(\xi), \quad \Gamma = \Gamma_0.$$

Then, the equations of motion become

$$\begin{gathered} -p_x + u_{\xi\xi} = 0, \qquad -p_y = 0, \qquad -p_\xi = 0, \\ \mathrm{Pr}^{-1}\theta_{\xi\xi} = u\theta_x, \qquad c_{\xi\xi} + \mathrm{Sr}\,(\theta_{xx} + \theta_{\xi\xi}) = 0; \end{gathered} \tag{1.7}$$

$$\xi = 0: \qquad u = 0, \quad \theta_\xi = 0, \quad c_\xi = 0; \tag{1.8}$$

$$\xi = 1: \quad u_\xi = -\mathrm{M}\theta_x, \quad p - p_1 = 0, \quad \theta_\xi + \mathrm{Bi}\,(\theta - \theta_1) + Q = 0, \quad c_\xi + \mathrm{Sr}\,\theta_\xi = 0, \quad c_\xi + \mathrm{Sr}\,\theta_\xi = -D_2(c - \Gamma), \tag{1.9}$$

where

$$\xi = \frac{z}{l}, \quad \mathrm{Pr} = \frac{\nu}{\chi}, \quad D_0 = \frac{D_s}{\nu}, \quad \mathrm{M} = \frac{æAl^2}{\rho\nu\chi}, \quad \mathrm{Bi} = \frac{\beta l}{\lambda},$$

$$\mathrm{M}_c = \frac{\gamma\Gamma_0 l}{\rho\nu^2}, \quad \mathrm{Sc} = \frac{\nu}{D}, \quad \mathrm{Sr} = \frac{k_T K_A \nu}{K_D \Gamma_0 \chi}, \quad D_1 = \frac{DlK_D}{K_A\nu}, \quad D_2 = \frac{lK_A}{D}.$$

Here Pr is the Prandtl number, M is the Marangoni number, Bi is the Biot number, M_c is the Marangoni concentration number, Sc is the Schmidt number, Sr is the Soret number, and x and y are dimensionless coordinates.

The solution of problem (1.7)–(1.9) that satisfies the closed flow condition is written as

$$u_0 = \frac{\mathrm{MPr}^{-1}}{2}\Big(\frac{3}{2}\,\xi^2 - \xi\Big), \qquad p_{0x} = \frac{3}{2}\,\mathrm{MPr}^{-1},$$

$$\theta_0 = -\frac{\mathrm{MPr}^{-1}}{12}\Big(\frac{3\xi^4+1}{4} - \xi^3\Big) - x\mathrm{Pr}^{-1}, \qquad c_0 = \frac{\mathrm{Sr}\,\mathrm{MPr}^{-1}}{12}\Big(\frac{3\xi^4+1}{4} - \xi^3\Big) + \Gamma_0. \tag{1.10}$$

System (1.7)–(1.9) also has a shear (Couette) solution but it is not considered here.

We study the stability of the quadratic or (as it is called) return flow (1.10). In the absence of surfactants and volume concentration, the stability of an analog of solution (1.10) is studied in [6].

In the following, we restrict ourselves to only thermocapillary Instability, ignoring surface waves. The free surface is considered rigid.

We seek a solution of problem (1.1)–(1.6) in the form $u = u_0 + U$, $v = V$, $w = W$, $p = p_0 + P$, $\theta = \theta_0 + T$, $c = c_0 + S$, and $\Gamma = \Gamma_0 + G$. Here U, V, and W is the velocity perturbations, P is the pressure, T is the temperature, S is the volume concentration, and G is the surface concentration. We assume that the perturbed quantities have the form

$$(U, V, W, P, T, S, G) = (U(\xi), V(\xi), W(\xi), P(\xi), T(\xi), S(\xi), G)\exp\,[i(\alpha x + \beta y - C\tau)], \tag{1.11}$$

where α and β are the wave numbers in the x and y directions, respectively, τ is dimensionless time, and $C = C_r + iC_i$ is the complex decrement.

Substitution of (1.11) into Eqs. (1.1)–(1.6) yields the following system of equations for the perturbations [8]

$$U_{\xi\xi} + a_1 U = i\alpha P + u_{0\xi} W, \qquad V_{\xi\xi} + a_1 V = i\beta P, \qquad W_{\xi\xi} + a_1 W = P_\xi,$$

$$i\alpha U + i\beta V + W_\xi = 0, \qquad T_{\xi\xi} + a_2 T = \mathrm{Pr}\theta_{0x} U + \mathrm{Pr}\theta_{0\xi} W, \tag{1.12}$$

$$S_{\xi\xi} + a_3 T = \mathrm{Sc}C_{0\xi} W - \mathrm{Sr}\,(T_{\xi\xi} - (\alpha^2+\beta^2)T),$$

where $a_1 = -(\alpha^2+\beta^2) - i\alpha u_0 + iC$, $a_2 = -(\alpha^2+\beta^2) - i\alpha\mathrm{Pr}u_0 + i\mathrm{Pr}C$, $a_3 = -(\alpha^2+\beta^2) - i\alpha\mathrm{Sc}u_0 + i\mathrm{Sc}C$,

$$\xi = 0: \qquad U = V = W = T_\xi = S_\xi = 0; \tag{1.13}$$

$$\xi = 1: \qquad U_\xi + i\alpha W = -i\alpha \mathrm{M}T - i\alpha \mathrm{M}_c G, \quad V_\xi + i\beta W = -i\beta \mathrm{M}T - i\beta \mathrm{M}_c G,$$

$$W = 0, \quad T_\xi + \mathrm{Bi}\,T = 0,$$

$$[-iC + D_0(\alpha^2+\beta^2) + i\alpha u_0]G + i\alpha U + i\beta V = -D_1(S_\xi + \mathrm{Sr}\,T_\xi), \tag{1.14}$$

$$S_\xi + \mathrm{Sr}\,T_\xi = -D_2(S - G).$$

2. Let us consider the case of monotonic perturbations ($C_r = 0$). We assume that the perturbations are plane ($\alpha = 0$) and restrict ourselves to constructing the neutral curves ($C_i = 0$).

In this case, problem (1.11)–(1.13) is considerably simplified and becomes

$$U_{\xi\xi} - \beta^2 U = u_{0\xi} W, \qquad V_{\xi\xi} - \beta^2 V = i\beta P,$$

$$W_{\xi\xi} - \beta^2 W = \mathrm{Pr}_\xi, \qquad i\beta V + W_\xi = 0,$$

$$T_{\xi\xi} - \beta^2 T = \mathrm{Pr}\theta_{0x} U + \mathrm{Pr}\theta_{0\xi} W, \tag{2.1}$$

$$S_{\xi\xi} - \beta^2 S = (\mathrm{Sc}C_{0\xi} - \mathrm{Sr}\,\mathrm{Pr}\theta_{0\xi})W - \mathrm{Sr}\,\mathrm{Pr}\theta_{0x} U;$$

$$\xi = 0: \qquad U = V = W = T_\xi = S_\xi = 0; \tag{2.2}$$

$$\xi = 1: \qquad U_\xi = 0, \quad W = 0, \quad V_\xi = -i\beta \mathrm{M}T - i\beta \mathrm{M}_c G, \quad T_\xi + \mathrm{Bi}\,T = 0,$$

$$D_0\beta^2 G + i\beta V = -D_1(S_\xi + \mathrm{Sr}\,T_\xi), \quad S_\xi + \mathrm{Sr}\,T_\xi = -D_2(S - G). \tag{2.3}$$

Solving system (2.1) subject to boundary conditions (2.2) and (2.3), we obtain the equation whose solution gives an explicit expression for the neutral Marangoni numbers in the case of plane monotonic perturbations:

$$\mathrm{M}^2\Big\{-\frac{A_1}{32\mathrm{Pr}\beta^3}-\frac{A_2}{32\beta}+\frac{\cosh\beta}{\beta\sinh\beta+\mathrm{Bi}\cosh\beta}\Big[\frac{1}{16\mathrm{Pr}\beta^2}\Big(A_3+\frac{\mathrm{Bi}}{2\beta}A_1\Big)+\frac{1}{8\beta}\Big(A_4+\frac{\mathrm{Bi}}{4}A_2\Big)\Big]$$

$$-A_5\frac{\beta}{\beta\sinh\beta+\mathrm{Bi}\cosh\beta}\Big\}+\mathrm{M}\Big\{-\frac{\mathrm{M}_cD_1D_2\,\mathrm{Sr}}{A_6}A_7-\frac{D_1D_2\cosh\beta\mathrm{M}_c}{\beta A_6}\Big[\frac{\mathrm{Sr}\,\beta\sinh\beta}{\beta\sinh\beta+\mathrm{Bi}\cosh\beta}$$

$$\times\Big(\frac{1}{16\mathrm{Pr}\beta^2}\Big(A_3+\frac{\mathrm{Bi}}{2\beta}A_1\Big)+\frac{1}{8\beta}\Big(A_4+\frac{\mathrm{Bi}}{4}A_2\Big)\Big)+\frac{\mathrm{Sr}\,\mathrm{Sc}}{8\mathrm{Pr}\beta}A_4\Big]-A_5\frac{D_1D_2\,\mathrm{Bi}\,\mathrm{M}_c\cosh\beta\,\mathrm{Sr}}{A_6(\beta\sinh\beta+\mathrm{Bi}\cosh\beta)}$$

$$+\frac{D_1D_2\mathrm{M}_c\sinh\beta\,\mathrm{Sr}}{A_6}\Big(\frac{1}{32\mathrm{Pr}\beta^3}A_1+\frac{1+\mathrm{Sc}\mathrm{Pr}^{-1}}{32\beta}A_2\Big)\Big\}$$

$$+\frac{1}{\beta^2}\Big(\cosh\beta-\frac{\beta}{\sinh\beta}\Big)+\frac{\mathrm{M}_c(\beta\sinh\beta+D_2\cosh\beta)}{2\beta A_6}\Big(\frac{\sinh\beta}{\beta}-\frac{\beta}{\sinh\beta}\Big)=0, \qquad (2.4)$$

where

$$A_1=-\frac{15\cosh\beta}{2\beta^3}-\frac{5\sinh\beta}{6}+\frac{15\cosh^2\beta}{2\beta^2\sinh\beta}+\frac{2+\cosh^2\beta}{\sinh\beta}-\frac{8\sinh\beta}{\beta^2};$$

$$A_2=\frac{3\cosh\beta}{\beta^5}+\frac{\cosh\beta}{15\beta}-\frac{3}{\beta^4\sinh\beta}+\frac{1}{10\sinh\beta}-\frac{1}{2\beta^2\sinh\beta};$$

$$A_3=-\frac{15\cosh\beta}{4\beta^4}+\frac{\sinh\beta}{4\beta^3}+\frac{\cosh\beta}{12}+\frac{\beta\sinh\beta}{12}+\frac{7\cosh\beta}{4\beta^2}+\frac{9}{4\beta^3\sinh\beta}-\frac{\beta\sinh^2\beta}{12\sinh\beta}$$

$$+\frac{1}{4\beta\sinh\beta}+\frac{1}{\beta\sinh\beta}+\frac{3\cosh^2\beta}{2\beta^3\sinh\beta}+\frac{1}{\cosh\beta}-\frac{2\sinh^2\beta}{\beta^2\cosh\beta}-\frac{7\sinh\beta}{2\beta^3};$$

$$A_4=\frac{3\cosh\beta}{4\beta^5}+\frac{\sinh\beta}{60}-\frac{3}{4\beta^4\sinh\beta}+\frac{\cosh\beta}{4\beta^3}+\frac{\cosh\beta}{4\beta^3}-\frac{3}{8\beta^2\sinh\beta};$$

$$A_5=\frac{1}{16\mathrm{Pr}\beta^3}\Big(-\frac{15}{4\beta^4}+\frac{15\cosh\beta}{4\beta^3\sinh\beta}-\frac{3}{2\beta^2}+\frac{1}{\beta\sinh\beta\cosh\beta}-\frac{2\sinh\beta}{\beta^3\cosh\beta}\Big)+\frac{1}{8\beta^2}\Big(\frac{3}{4\beta^5}-\frac{3\cosh\beta}{4\beta^4\sinh\beta}+\frac{5}{8\beta^3}\Big);$$

$$A_6=(D_0\beta^2+D_1D_2)\sinh\beta+D_2D_0\beta\cosh\beta;$$

$$A_7=\frac{1}{16\mathrm{Pr}\beta^3}\Big(-\frac{15}{4\beta^4}+\frac{15\cosh\beta}{4\beta^3\sinh\beta}-\frac{3}{2\beta^2}+\frac{1}{\beta\sinh\beta\cosh\beta}-\frac{2\sinh\beta}{\beta^3\cosh\beta}\Big)-\frac{1}{8\beta^2}\Big(1+\frac{\mathrm{Sc}}{\mathrm{Pr}}\Big)\Big(\frac{3}{4\beta^5}-\frac{3\cosh\beta}{4\beta^4\sinh\beta}+\frac{5}{8\beta^3}\Big).$$

Equation (2.4) is quadratic in the Marangoni number; therefore, in the plane case there are two neutral curves that describe the flow stability boundary for monotonic perturbations.

3. Problem (1.12)–(1.14) is solved numerically for arbitrary perturbations using the orthogonalization method. The analytically obtained neutral curves for monotonic perturbations are used as tests in the calculations. In addition, the solutions of Eq. (2.4) serve as initial approximations in the calculations of three-dimensional perturbations.

The existence of Eq. (2.4) implies that the presence of impurities in the liquid leads to instability with respect to monotonic perturbations in the plane case, too ($\alpha = 0$). In the absence of impurities, such instability for the return flow is not observed [6]. Typical neutral curves (curves 1 and 2) for monotonic perturbations are plotted in Fig. 1 for $D_0 = 10^{-4}$, $D_1 = 30$, $D_2 = 10^3$, Sr = 10, Sc = 10, $\mathrm{M}_c = 5$, Bi = 0, and Pr = 0.016. According to [6], the perturbations corresponding to these neutral curves are induced by the thermocapillary instability mechanism due to nonuniform heating of the free surface and are stationary longitudinal rolls whose axes are directed downstream. The region of instability is between the neutral curves.

In addition, instability of the main flow can arise from the hydrodynamic perturbations due to motion in the liquid. Such perturbations are manifested as waves which propagate in both directions perpendicular to the main

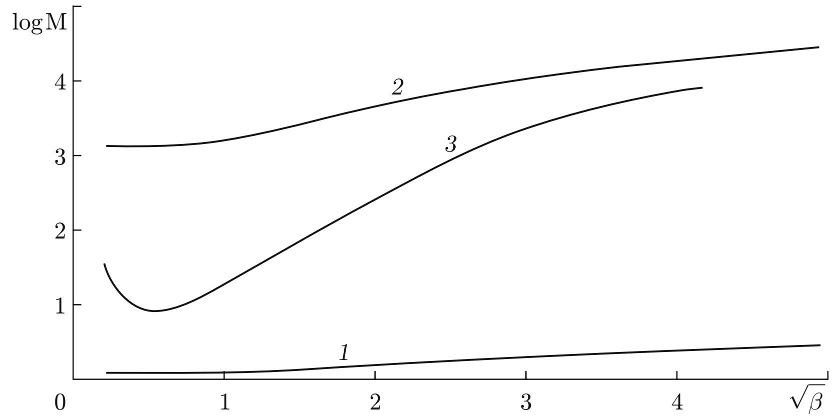

Fig. 1. Neutral curves for plane perturbations for $\alpha = 0$.

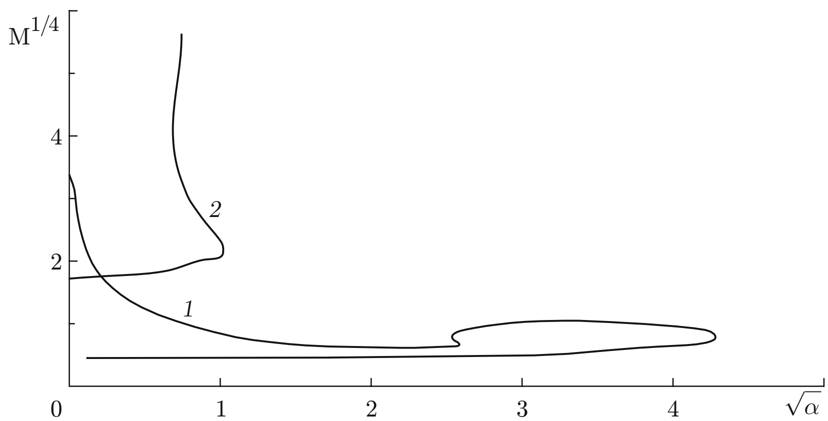

Fig. 2. Neutral curves for volume perturbations for $\beta = 0.3$.

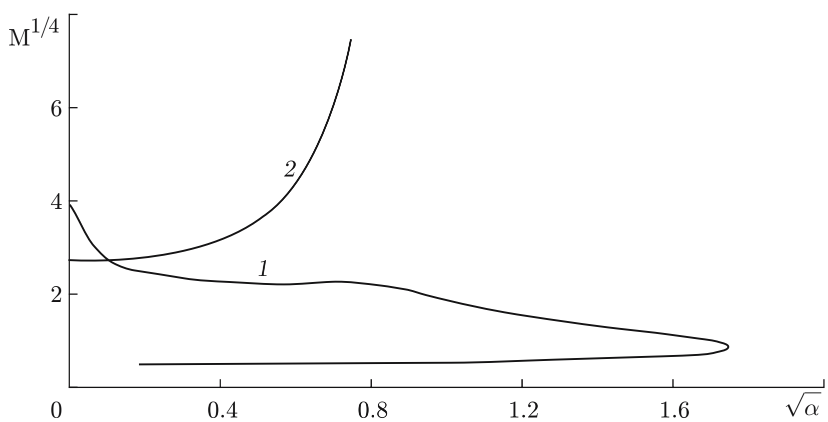

Fig. 3. Neutral curves for volume perturbations for $\beta = 2$.

flow. The neutral curve corresponding to hydromagnetic instability for $\alpha = 0$ is also given in Fig. 1 (curve 3). In contrast to the thermocapillary mode, the hydrodynamic perturbations grow in an oscillatory manner. The region of instability is below the neutral curve. Thus, the flow stability boundary with respect to plane perturbations is described by curve 1.

Let us consider volume perturbations. Figures 2 and 3 give cross sections of the neutral surfaces for $\beta = 0.3$ and $\beta = 2$, respectively, for thermocapillary (curve 1) and hydrodynamic (curve 2) perturbations. The values of the remaining parameters are the same as in Fig. 1. The lower edge of the given neutral curves for thermocapillary perturbations for $\alpha = 0$ coincides with curve 1 in Fig. 1, and the upper edge with curve 2 in Fig. 1. We note that for $\alpha = 0$, the thermocapillary perturbations cease to be monotonic and become oscillating. The region of instability in Figs. 2 and 3 is inside the neutral curve 1 and is lower than curve 2. As the wave number β increases, the region of instability for thermocapillary perturbations is shifted toward large Marangoni numbers. In this case, the region of instability increases insignificantly on the α axis and the motion is always stable against short-wavelength perturbations. Hydrodynamic perturbations play a leading role in flow instability only in the region of relatively long waves. In the remaining range of wavenumbers, the stability boundary is described by curve 1, which corresponds to thermocapillary perturbations. The calculation results given in the figures show that the plane perturbations at a wavenumber $\alpha = 0$ are the most dangerous for the stability of the return motion (1.10).

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REFERENCES

1. J. R. A. Pearson, "On convection cells induced by surface tension," *J. Fluid Mech.*, **5**, No. 4, 489–500 (1958).
2. J. C. Berg and A. Acrivos, "The effect of surface active agents on convection cells induced by surface tension," *Chem. Eng. Sci.*, **20**, No. 8, 737–745 (1965).
3. H. J. Palmer and J. C. Berg, "Hydrodynamic stability of surfactant solutions heated from below," *J. Fluid Mech.*, **51**, Part 2, 385–402 (1972).
4. E. A. Ryabitskii, "Occurrence of thermocapillary motion in a flat layer with allowance for the Soret effect," *Izv. Ross. Akad. Nauk, Ser. Mekh. Zhidk. Gaza*, No. 3, 3–9 (2000).
5. B. N. Goncharenko and A. L. Urintsev, "Stability of motion caused by thermocapillary forces," *J. Appl. Mech. Tech. Phys.*, No. 6, 94–98 (1977).
6. M. K. Smith and S. H. Davis, "Instabilities of dynamic thermocapillary liquid layers. Part 1. Convective instabilities," *J. Fluid Mech.*, **132**, No. 7, 119–144 (1983).
7. M. K. Smith and S. H. Davis, "Instabilities of dynamic thermocapillary liquid layers. Part 2. Surface wave instabilities," *ibid.*, pp. 145–162.
8. V. K. Andreev, V. E. Zakhvataev, and E. A. Ryabitskii, *Thermocapillary Instability* [in Russian], Nauka, Novosibirsk (2000).